7 - Products

We already defined unordered collections ("sets"). We can also define ordered collections, called sequences, lists, or tuples. This is useful: RGB != RBG.

Definition. A **tuple** (aka sequence or list) is an ordered collection of objects, typically called components or entries. When the number of objects in the collection is 2, 3, 4, or n, the tuple is called an (ordered) pair, triple, quadruple, or, n-tuple, respectively.

Definition. The **Cartesian product** of two sets A and B is the set A x B defined by

$$A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$$

containing all ordered pairs where the first component comes from A and the second component comes from B.

Example 1. List all orders of ice cream with strawberry (s), Ube (u), and vanilla (v) flavors and come in bowl (b) or cone (c).

{s,u,v} x {b,c}: (s,b), (u,b), (v, b),
(s,b), (u,c), (v, c)

Example 2. The **RGB color space** is the set

$$\{0,1,...,255\} \times \{0,1,...,255\} \times \{0,1,...,255\} = \{0,1,...,255\}^3$$

An **RGB color** is a 3-tuple or triple (r, g, b).

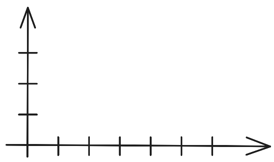
Definition. For a set S and $n \in \mathbb{N}$, we write $S^n = \underbrace{S \times S \times \cdots \times S}_{n \text{ times}}$

Example 3. The **Cartesian plane** is the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ which contains an element like (2,3).

Example 4. Graph (a) {2,4,6} x {1,3} (b) [2,6] x [1,3] on the Cartesian plane.

(a)

(b)



An element of S^n is also called a **string** of length n in the **alphabet** S.
An **n-bit string** is a string of length n in alphabet {0,1}.

Convention: omit parentheses & commas when writing strings:

(T,U,P,L,E) = "TUPLE" ∈ $\{A, B, ..., Z\}^8$ and "11010011" ∈ $\{0, 1\}^{11}$

Product Rule. Let A and B be sets. Then |A x B| = |A||B|.

Example 1: the # of possible ice cream orders is: 3 x 2 = 6.
Example 2: the # of RGB colors is 256 x 256 x 256.

Example 6. US texts come from 10-digit phone numbers, 5-digit short code, or 6-digit short codes. Short codes cannot start with 0 or 1. How many 5-digit short codes are possible?

Answer: $\{2,3,4,...,9\} \times \{0,1,2,3,4,...,9\}^4$ has size $8 \times 10^4 = 80000$.

Example 7. IP addresses are 32-bit strings. How many IP addresses are possible?

Answer: $\{0,1\}^{32}$ has size 2^{32} ≈ 4 billion. The world already exceed this many devices. That's why in 2000s we have switched to IPv6 standard which uses 128-bit strings for IP addresses.

Example 8. How many subsets of {a, b, c} are there?

Use \`ipconfig\` or \`ping\` or \`nslookup\` in terminal to demonstrate these IP addresses.

Answer: (Brute-force) A subset uses some elements of the set:
{a,b,c}, {a,b}, {a,c}, {b,c}, {a}, {b}, {c}, {} so 8.

Example 9. How many subsets of {a, b, c, d, e} are there?

Answer: We use Yes-No string encoding to determine if an element belongs to a set:
YYNYN <-> {a,b,d}
NNNNN <-> {} (empty set)
Overall, there are 2^5=32 such strings, so there are 32 subsets of {a,b,c,d,e}.

Example 10. How many 3-digit numbers have only even numbers in its digits? Example: 2420. Non-examples: 2421, 0264 (0264=264 is a 3-digit number).

Answer: (__, __, __, __)

4 options, 5 options, 5 options, 5 options

Example 11. In a race, how many ways can Ann, Ben, Cat, & Dan be awarded 1st & 2nd places?

Answer:
{1st, 2nd, 3rd, 4th options} x {1st, 2nd, 3rd remaining options} x {1st, 2nd remaining options}

Definition. "n factorial" means n! = n x (n-1) x (n-2) x (n-3) x ... x 3 x 2 x 1.

so: 0! = 1 (convention), 1! = 1, 2! = 2x1 = 2, 3! = 3x2x1 = 6.

Example 12. How many ways can the letters of the word "RANDOM" be rearranged?

Answer: (__, __, __, __, __, __) so the answer is 6! = 6x5x4x3x2x1 = 720.

6 options, 5 options, 4 options, 3 options, 2 options, 1 option

> A **k-permutation** is an ordered selection of k things from n things, with no repeats. The number of k-permutations is P(n,k) = n!/(n-k)!

Example 11: P(4,2) = 4!/(4-2)! = 4!/2! = (4x3x2x1)/(2x1) = 4x3.
Example 12: P(6,6) = 6!/(6-6)! = 6!/0! = 6!

Example 13. How many ways can 10 books on one bookshelf be arranged?

Answer: P(10,10) = 10!

Optional Homework due March 4th or 5th. Show your work. Answer without work receives no credit.
1. Draw the Venn diagram of ~(A∪B). (Review)
2. Draw the Venn diagram of (A∩(~B))∪C. (Review)
4. List the elements of the set $\{n \in \mathbb{N} : n \equiv 0 \pmod 3\} \cap [1000, 1010]$
5. A coin is tossed 3 times. How many possible sequences of heads and tails can result?
6. A dice is rolled 2 times. How many possible sequences of 1,2,3,4,5,6 can result?
7. How many 4-digit numeric passcodes use only even digits? Examples: "0240", 8228","2026".
8. How many 3-digit hexadecimals are there in which the first digit is E or F?
9. A standard license plate consists of three letters followed by four digits, like "JRB-4412" and "MMX-8901". How many different standard license plates are possible?
8. How many subsets of {1, 2, 3, 4, 5, 6, 7} have {2, 3, 5} as a subset?
9. How many ways can 1st, 2nd, 3rd places be awarded to 10 runners in a race? (Assume no ties.)
10. How many bitstrings of length 5 start with a 1 or end with a 1?
11. How many 5-digit ternary numbers start with a 2 and end with a 2?